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A Lindström theorem in many-valued modal logic over a finite MTL-chain

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Received 2 July 2018; received in revised form 5 March 2019; accepted 5 March 2019

Abstract

We consider a modal language over crisp frames and formulas evaluated on a finite MTL-chain (a linearly ordered commutative integral residuated lattice). We first show that the basic modal abstract logic with constants for the values of the MTL-chain is the maximal abstract logic satisfying Compactness, the Tarski Union Property and strong invariance for bisimulations. Finally, we improve this result by replacing the Tarski Union Property by a relativization property.

Keywords: Many-valued modal logic; Fuzzy logic; MTL-chains; Residuated lattices; Bisimulation; Lindström Theorem

0. Introduction

Mathematical Fuzzy Logic (MFL) is well-established line of research in contemporary mathematical logic (Cintula et al. eds. [9]). The model theory of first order predicate fuzzy logics has been developed in MFL beginning with Hájek [18]. The first-order structures which are the center of the topic are also known as *weighted structures* in computer science (cf. Horčík et al. [21]). Some recent references are Hájek and Cintula [19], Cintula and Noguera [10], Dellunde and Esteva [14], Dellunde et al. [15], Dellunde [11], [13] and [12].

In this paper, our setting will be that of modal logics based on a finite MTL-chain. These logics have generated some literature, in particular, our point of inspiration is Bou et al. [5]. The interest in this kind of system consists in mixing modal reasoning with graded reasoning. Most of the work on many-valued modal logic has been devoted to problems having to do with axiomatizability and computability (recent examples are Vidal et al. [27] and Caicedo et al. [6]). However, the problem of the expressivity of modal languages in a many-valued setting has only been recently

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https://doi.org/10.1016/j.fss.2019.03.002
0165-0114/© 2019 Published by Elsevier B.V.

Please cite this article in press as: G. Badia, G. Olkhovikov, A Lindström theorem in many-valued modal logic over a finite MTL-chain, Fuzzy Sets Syst. (2019), https://doi.org/10.1016/j.fss.2019.03.002

tackled in Marti and Metcalfe [23] and Bílková and M. Dostál [4], focusing on the Hennessy-Milner property. Much work still remains to be done in this sense, though.

We will establish that the basic modal abstract logic with constants for the values of the chain is the maximal abstract logic satisfying Compactness, the Tarski Union Property and strong invariance for bisimulations. This result follows a well-known pattern set by the characterization of classical first-order logic by Per Lindström (published as Lindström [22]) which has had a great deal of repercussion in contemporary logic.

Lindström's theorem single-handedly started a new area of research known as *abstract* or *soft* model theory (cf. Barwise [1] and Barwise and Feferman [2]). These two adjectives are used since, when working in this field, one finds themself using "only very general properties of the logic, properties that carry over to a large number of other logics" (Barwise [1], p. 225).

Naturally, there is no reason why one could not do non-classical abstract model theory as well, this time focusing on some non-classical logic and its structures. An interesting survey of this much less developed field can be found in García-Matos and Väänänen [17]. The most prominent work has been done in the case of modal logic. A succession of increasingly improved results were obtained by a number of authors such as de Rijke [25], van Benthem [26], Otto and Piro [24] and, more recently, Enqvist [16]. The first Lindström-type result presented below uses a variant of technique employed in Otto and Piro [24] for classical modal logic, whereas the second outlines variants of the techniques from van Benthem [26].

The layout of the article is as follows. In Section 1 we define the semantics of the many-valued logic at hand and define some basic model-theoretic notions like embedding and substructure. In Section 2 we instantiate the framework of abstract model theory for the extensions of the logic defined in Section 1. In Section 3 we do the same for the notion of bisimulation and recall some related properties of the logic at hand, e.g. Hennessy-Milner property and strong invariance for bisimulations. In Section 4 we introduce Tarski Union Property and briefly show that it holds for the logic defined in Section 1. Section 5 then proves the main result of the paper, i.e. that no compact extension of our many-valued logic displays both Tarski Union Property and strong invariance for bisimulations. In Section 6, we show that we can replace the Tarski Union Property by a certain relativization property. Finally, Section 7 sums up the work.

1. Semantic preliminaries

The algebraic framework is going to be that of MTL-algebras (Cintula et al. [8]), that is, algebraic structures of the form $\boldsymbol{A} = \langle A, \wedge^{\boldsymbol{A}}, \vee^{\boldsymbol{A}}, \&^{\boldsymbol{A}}, \rightarrow^{\boldsymbol{A}}, \overline{0}^{\boldsymbol{A}}, \overline{1}^{\boldsymbol{A}}\rangle$ such that

- $\langle A, \wedge^{\boldsymbol{A}}, \vee^{\boldsymbol{A}}, \overline{0}^{\boldsymbol{A}}, \overline{1}^{\boldsymbol{A}}\rangle$ is a bounded lattice,
- $\langle A, \&^{\boldsymbol{A}}, \overline{1}^{\boldsymbol{A}}\rangle$ is a commutative monoid,
- for each $a, b, c \in A$, we have:

$$a \&^{\boldsymbol{A}} b \leq c \quad \text{iff} \quad b \leq a \rightarrow^{\boldsymbol{A}} c, \qquad \text{(res)}$$

$$(a \rightarrow^{\boldsymbol{A}} b) \vee^{\boldsymbol{A}} (b \rightarrow^{\boldsymbol{A}} a) = \overline{1}^{\boldsymbol{A}} \qquad \text{(prelin)}$$

$\boldsymbol{A}$ is called a MTL-*chain* if its underlying lattice is linearly ordered. Henceforth, we will confine our attention to algebras which are chains of such kind. For the remainder of the paper, we will be working with a fixed finite MTL-chain $\boldsymbol{A}$.

The restriction to finite fixed MTL-algebras comes from Bou et al. [5], since we had to appeal to the results there to establish our Compactness property (in particular to Theorem 4.22 of that paper), so we refer the reader to Bou et al. [5] for an explanation. Moreover, restricting ourselves to chains is actually very much in the spirit of fuzzy logic Běhounek and Cintula [3]. A more technical reason is the proof of Theorem 9, where we make explicit use of this. However, we could relax the restriction a little bit by requiring instead that our algebras have a unique co-atom (this idea comes from Bou et al. [5]), say a, and requiring that for every element a of the algebra $a \leq b$ or $b \leq a$. The results in this paper work in this slightly more relaxed framework as well.

Consider modal language $\mathcal{L}_0(\tau)$ based on a set of propositional variables τ (also called signature) and the set of connectives $\mathsf{Con}^{A} = \{\wedge, \vee, \rightarrow, \&, \Box, \Diamond\} \cup \{\overline{b} \mid b \in A\}$ where each $\overline{b}$ is a truth-value constant denoting $b \in A$. When

$b = \overline{1}^{\boldsymbol{A}}$ or $b = \overline{0}^{\boldsymbol{A}}$, then we write $\overline{b}$ as $\overline{1}$ or $\overline{0}$ respectively. Since MTL-chains include at least two elements, $\overline{1}^{\boldsymbol{A}}$ must have an immediate predecessor in terms of the lattice order. We will reserve the notation a for this predecessor and $\overline{a}$ for its corresponding constant in $\mathsf{Con}^{\boldsymbol{A}}$. Note that $\overline{a}$ is in fact a meta-constant; for instance, for some instantiations of $\boldsymbol{A}$, we may have $\overline{a} = \overline{0}$.

The $\boldsymbol{A}$-structures $\mathfrak{M}$ of signature τ (also called $(\boldsymbol{A}, \tau)$-models) will be of the form $\langle W^{\mathfrak{M}}, R^{\mathfrak{M}}, [\tau]^{\mathfrak{M}} \rangle$ where $W^{\mathfrak{M}}$ is a non-empty set of worlds, $R^{\mathfrak{M}}$ is a binary crisp $\boldsymbol{A}$-valued relation (takes values just in the set $\{\overline{1}^{\boldsymbol{A}}, \overline{0}^{\boldsymbol{A}}\}$) and $[\tau]^{\mathfrak{M}}$ is a collection of $\boldsymbol{A}$-valued unary relations $[p]^{\mathfrak{M}}$ (i.e., of functions of the form $[p]^{\mathfrak{M}} : W^{\mathfrak{M}} \longrightarrow A$) for each propositional variable $p \in \tau$. The pair $\langle W^{\mathfrak{M}}, R^{\mathfrak{M}} \rangle$ is also called a crisp frame. The results of this paper might be generalizable to frames with many-valued binary accessibility relations, however, since we are appealing to work from Marti and Metcalfe [23] where the setting is crisp frames, the possible generalization is left for future work.

Formulas in $\mathcal{L}_0(\tau)$ are evaluated at points $w \in W^{\mathfrak{M}}$, so it makes sense to speak of pointed $(\boldsymbol{A}, \tau)$-models of the form $\langle \mathfrak{M}, w \rangle$ since this kind of structures is the basic semantic unit of the subject. We define the *truth values* of the formulas as:

$$
\begin{aligned}
\|p\|^{\langle \mathfrak{M}, w \rangle} &= [p]^{\mathfrak{M}}(w) \quad (p \in \tau) \\
\|\overline{b}\|^{\langle \mathfrak{M}, w \rangle} &= b \quad (b \in A) \\
\|\overline{1}\|^{\langle \mathfrak{M}, w \rangle} &= \overline{1}^{\boldsymbol{A}} \\
\|\overline{0}\|^{\langle \mathfrak{M}, w \rangle} &= \overline{0}^{\boldsymbol{A}} \\
\|\circ(\varphi_1, \varphi_2)\|^{\langle \mathfrak{M}, w \rangle} &= \circ^{\boldsymbol{A}}(\|\varphi_1\|^{\langle \mathfrak{M}, w \rangle}, \|\varphi_2\|^{\langle \mathfrak{M}, w \rangle}) \quad (\circ \in \{\wedge, \vee, \rightarrow, \&\}) \\
\|\Box\varphi\|^{\langle \mathfrak{M}, w \rangle} &= \inf_{\leq_{\boldsymbol{A}}}\{\|\varphi\|^{\langle \mathfrak{M}, v \rangle} \mid R^{\mathfrak{M}}(w, v) = \overline{1}^{\boldsymbol{A}}\}, \\
\|\Diamond\varphi\|^{\langle \mathfrak{M}, w \rangle} &= \sup_{\leq_{\boldsymbol{A}}}\{\|\varphi\|^{\langle \mathfrak{M}, v \rangle} \mid R^{\mathfrak{M}}(w, v) = \overline{1}^{\boldsymbol{A}}\}.
\end{aligned}
$$

Following definitions in Dellunde [11] (Section 3.1), we introduce the following notion of embedding between the models defined above:

Definition 1. Let $\mathfrak{M}$ and $\mathfrak{N}$ be two $(\boldsymbol{A}, \tau)$-models. An injection $f : W^{\mathfrak{M}} \longrightarrow W^{\mathfrak{N}}$ is said to be an *embedding* from $\mathfrak{M}$ to $\mathfrak{N}$ if for every S (binary predicate) and T (unary predicate) in $\tau \cup \{R\}$ and every sequence $d_1, d_2 \in W^{\mathfrak{M}}$,

$$
\begin{aligned}
&S^{\mathfrak{M}}(d_1, d_2) = S^{\mathfrak{N}}(f(d_1), f(d_2)), \\
&T^{\mathfrak{M}}(d_1) = T^{\mathfrak{N}}(f(d_1)).
\end{aligned}
$$

Moreover, when f is surjective, we obtain the notion of *isomorphism*, and when f is the identity we obtain the notion of a *substructure*. All these notions can be extended to pointed models so that, e.g., for any $w \in W^{\mathfrak{M}}$, $\langle \mathfrak{M}, w \rangle$ is a substructure of $\langle \mathfrak{N}, w \rangle$ iff $\mathfrak{M}$ is a substructure of $\mathfrak{N}$.

Note that since in the Definition 1 f is a homomorphism, the condition for f holds not only for atomic but for all modal operators-free formulas.

2. Abstract modal logics and compactness of $\mathcal{L}_0^{\boldsymbol{A}}$

For a given finite MTL-chain $\boldsymbol{A}$ we will call the structure $\mathcal{L}_0^{\boldsymbol{A}} = \langle \mathcal{L}_0, \|\,\| \rangle$ the modal propositional logic based on $\boldsymbol{A}$. More generally we will call an abstract extension of $\mathcal{L}_0^{\boldsymbol{A}}$ any structure of the form $\mathcal{L}^{\boldsymbol{A}} = \langle \mathcal{L}_{\mathcal{L}}, \|\,\|_{\mathcal{L}} \rangle$ such that:

- $\mathcal{L}_{\mathcal{L}}$ maps every signature τ to a set $\mathcal{L}_{\mathcal{L}}(\tau)$ (called the set of $\mathcal{L}(\tau)$-formulas) such that:
  - If $\tau \subseteq \tau'$, then $\mathcal{L}_{\mathcal{L}}(\tau) \subseteq \mathcal{L}_{\mathcal{L}}(\tau')$.
  - (Occurrence). If $\varphi \in \mathcal{L}_{\mathcal{L}}(\tau)$, then there is a finite $\tau_\varphi \subseteq \tau$ such that for every signature τ', $\varphi \in \mathcal{L}_{\mathcal{L}}(\tau')$ iff $\tau_\varphi \subseteq \tau'$.
  - (Closure). Every $\mathcal{L}_{\mathcal{L}}(\tau)$ contains τ as a subset and is closed for the connectives in $\mathsf{Con}^{\boldsymbol{A}}$.
- $\|\,\|_{\mathcal{L}}$ is a function which maps every pair $\langle \varphi, \langle \mathfrak{M}, w \rangle \rangle$ to an element of $\boldsymbol{A}$, where, for some signature τ, $\varphi \in \mathcal{L}_{\mathcal{L}}(\tau)$ and $\langle \mathfrak{M}, w \rangle$ is a pointed $(\boldsymbol{A}, \tau)$-model. Moreover, $\|\,\|_{\mathcal{L}}$ is assumed to extend the semantics of $\mathcal{L}_0^{\boldsymbol{A}}$ (i.e., it will respect the interpretation of connectives in $\mathsf{Con}^{\boldsymbol{A}}$ and propositional variables) and satisfy the following conditions:

- (Isomorphism). Whenever $\varphi \in \mathcal{L}_{\mathcal{L}}(\tau)$, $\langle \mathfrak{M}, w\rangle$ and $\langle \mathfrak{N}, v\rangle$ are pointed $(\boldsymbol{A}, \tau)$-models, and f is an isomorphism between $\langle \mathfrak{M}, w\rangle$ and $\langle \mathfrak{N}, v\rangle$, then

$$\|\varphi\|_{\mathcal{L}}^{\langle \mathfrak{N}, v\rangle} = \|\varphi\|_{\mathcal{L}}^{\langle \mathfrak{M}, w\rangle}.$$

- (Expansion). If τ, τ' are two signatures such that $\tau \subseteq \tau'$ and $\varphi \in \mathcal{L}_{\mathcal{L}}(\tau)$, $\langle \mathfrak{M}, w\rangle$ is a pointed $(\boldsymbol{A}, \tau')$-model and $\mathfrak{M} \restriction \tau$ is the reduct of $\mathfrak{M}$ to τ, then

$$\|\varphi\|_{\mathcal{L}}^{\langle \mathfrak{M}, w\rangle} = \|\varphi\|_{\mathcal{L}}^{\langle \mathfrak{M}\restriction\tau, w\rangle}.$$

It is easy to see that $\mathcal{L}_0^{\boldsymbol{A}}$ itself turns out to be its own abstract (improper) extension under this definition.

For an arbitrary abstract extension $\mathcal{L}^{\boldsymbol{A}}$ of $\mathcal{L}_0^{\boldsymbol{A}}$, we say that a pointed $(\boldsymbol{A}, \tau)$-model $\langle \mathfrak{M}, w\rangle$ $\mathcal{L}^{\boldsymbol{A}}$-*satisfies* a formula φ, if $\|\varphi\|_{\mathcal{L}}^{\langle \mathfrak{M}, w\rangle} = \overline{1}^{\boldsymbol{A}}$. We will put this in symbols many times as $\mathfrak{M}, w \models_{\mathcal{L}} \varphi$. For a set of formulas Φ, we write $\|\Phi\|_{\mathcal{L}}^{\langle \mathfrak{M}, w\rangle} = 1^{\boldsymbol{A}}$ (or $\mathfrak{M}, w \models_{\mathcal{L}} \Phi$) if $\|\varphi\|_{\mathcal{L}}^{\langle \mathfrak{M}, w\rangle} = 1^{\boldsymbol{A}}$ ($\mathfrak{M}, w \models_{\mathcal{L}} \varphi$) for every $\varphi \in \Phi$. We say that $\mathfrak{M}$ is an $\mathcal{L}$-*model of a set of formulas* Φ if $\|\varphi\|_{\mathcal{L}}^{\langle \mathfrak{M}, w\rangle} = 1^{\boldsymbol{A}}$ for any $\varphi \in \Phi$ and some $w \in W^{\mathfrak{M}}$. In a similar fashion, we will sometimes abbreviate the fact that $R^{\mathfrak{M}}(w, v) = \overline{1}^{\boldsymbol{A}}$ by writing simply $R^{\mathfrak{M}}(w, v)$.

Furthermore, let $\mathfrak{M}$ be an $(\boldsymbol{A}, \tau)$-model and let $w \in W^{\mathfrak{M}}$. A set $\Phi \subseteq \mathcal{L}_{\mathcal{L}}(\tau)$ we will call an $\mathcal{L}_{\Diamond}^{\boldsymbol{A}}$-type of $\langle \mathfrak{M}, w\rangle$ iff for every finite $\Phi_0 \subseteq \Phi$ there exists a $v \in W^{\mathfrak{M}}$ such that $R^{\mathfrak{M}}wv$ and $\mathfrak{M}, v \models_{\mathcal{L}} \Phi_0$. Model $\mathfrak{M}$ is then called $\mathcal{L}^{\boldsymbol{A}}$-*saturated*, iff for every $w \in W^{\mathfrak{M}}$ and every $\mathcal{L}_{\Diamond}^{\boldsymbol{A}}$-type Φ of $\langle \mathfrak{M}, w\rangle$ there exists a $v \in W^{\mathfrak{M}}$ such that $R^{\mathfrak{M}}wv$ and $\mathfrak{M}, v \models_{\mathcal{L}} \Phi$. It follows from the Closure that for every abstract extension $\mathcal{L}^{\boldsymbol{A}}$ of $\mathcal{L}_0^{\boldsymbol{A}}$, every $(\mathcal{L}_0^{\boldsymbol{A}})_{\Diamond}$-type is also a $\mathcal{L}_{\Diamond}^{\boldsymbol{A}}$-type, and hence every $\mathcal{L}^{\boldsymbol{A}}$-saturated model is also $\mathcal{L}_0^{\boldsymbol{A}}$-saturated.

We will say that for $\Phi \cup \{\varphi\} \subseteq \mathcal{L}_{\mathcal{L}}(\tau)$ φ 1-follows from Φ in $\mathcal{L}$, and write $\Phi \models_{\mathcal{L}} \varphi$ iff for any $(\boldsymbol{A}, \tau)$-model $\mathfrak{M}$ and any $w \in W^{\mathfrak{M}}$, $\langle \mathfrak{M}, w\rangle \models_{\mathcal{L}} \Phi$ implies that $\langle \mathfrak{M}, w\rangle \models_{\mathcal{L}} \varphi$. We will say that $\varphi, \psi \in \mathcal{L}_{\mathcal{L}}(\tau)$ are *1-equivalent* in $\mathcal{L}^{\boldsymbol{A}}$ iff both $\varphi \models_{\mathcal{L}} \psi$ and $\psi \models_{\mathcal{L}} \varphi$. Under the same assumptions, we define $Th_{\mathcal{L}}(\mathfrak{M}, w)$, an $\mathcal{L}$-*theory* of $\langle \mathfrak{M}, w\rangle$, as follows:

$$Th_{\mathcal{L}}(\mathfrak{M}, w) = \{\varphi \in \mathcal{L}_{\mathcal{L}}(\tau) \mid \mathfrak{M}, w \models_{\mathcal{L}} \varphi\}$$

We further define that given a pair $\mathcal{L}_1^{\boldsymbol{A}}$ and $\mathcal{L}_2^{\boldsymbol{A}}$ of abstract extensions of $\mathcal{L}_0^{\boldsymbol{A}}$, we say that $\mathcal{L}_2^{\boldsymbol{A}}$ extends $\mathcal{L}_1^{\boldsymbol{A}}$ and write $\mathcal{L}_1^{\boldsymbol{A}} \trianglelefteq \mathcal{L}_2^{\boldsymbol{A}}$ when for every signature τ and every $\varphi \in \mathcal{L}_{\mathcal{L}_1}(\tau)$ there exists a $\psi \in \mathcal{L}_{\mathcal{L}_2}(\tau)$ such that they are 1-equivalent for all pointed $(\boldsymbol{A}, \tau)$-models $\langle \mathfrak{M}, w\rangle$.

If both $\mathcal{L}_1^{\boldsymbol{A}} \trianglelefteq \mathcal{L}_2^{\boldsymbol{A}}$ and $\mathcal{L}_2^{\boldsymbol{A}} \trianglelefteq \mathcal{L}_1^{\boldsymbol{A}}$ hold, then we say that the abstract extensions $\mathcal{L}_1^{\boldsymbol{A}}$ and $\mathcal{L}_2^{\boldsymbol{A}}$ are *expressively equivalent* and write $\mathcal{L}_1^{\boldsymbol{A}} \simeq \mathcal{L}_2^{\boldsymbol{A}}$.

This notion of expressive equivalence between abstract logics in terms of 1-equivalence was first proposed in the fuzzy and many-valued setting by Hájek in [20]. It seems appropriate given that it connects to satisfaction and expressivity in a natural way. Moreover, for the case of fuzzy predicate logics, a Lindström theorem showing equivalence in this sense is the only one known in the literature (for the particular example of Łukasiewicz logic on the standard algebra in [0, 1]), a result due to Caicedo [7].

Note that $\mathcal{L}_0^{\boldsymbol{A}}$ is obviously $\trianglelefteq$ among its own abstract extensions, so that these extensions extend $\mathcal{L}_0^{\boldsymbol{A}}$ in the sense of the partial $\trianglelefteq$-order.

The following important fact about $\mathcal{L}_0^{\boldsymbol{A}}$ follows from theorem 4.22 (2) of Bou et al. [5] (note that, by Lemma 4.24 in that paper, our semantics coincides with theirs).

Proposition 1. *$\mathcal{L}_0^{\boldsymbol{A}}$ is compact, that is, every set of formulas that is finitely satisfiable is itself satisfiable.*

In the assumptions of the present paper, compactness for (1-)satisfiability implies also compactness for 1-consequence:

Corollary 2. *Assume that $\mathcal{L}_0^{\boldsymbol{A}} \trianglelefteq \mathcal{L}^{\boldsymbol{A}}$. If $\mathcal{L}^{\boldsymbol{A}}$ is compact, then, for every signature τ and every $\Phi \cup \{\varphi\} \subseteq \mathcal{L}_{\mathcal{L}}(\tau)$, such that $\Phi \models_{\mathcal{L}} \varphi$, there exists a finite $\Phi' \subseteq \Phi$ such that $\Phi' \models_{\mathcal{L}} \varphi$.*

Proof. We have $\Phi \models_{\mathcal{L}} \varphi$ iff $\Phi \cup \{\varphi \to \overline{a}\}$ is not (1-)satisfiable in $\mathcal{L}^{\boldsymbol{A}}$ iff, by Proposition 1, there exists a finite $\Phi' \subseteq \Phi$ such that $\Phi' \cup \{\varphi \to \overline{a}\}$ is not (1-)satisfiable in $\mathcal{L}^{\boldsymbol{A}}$ iff $\Phi' \models_{\mathcal{L}} \varphi$ for this Φ' □

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3. Bisimulations

In this section we recall the notion of a bisimulation in a many-valued setting (which appeared in Marti and Metcalfe [23]).

Definition 2. Two pointed $(\boldsymbol{A}, \tau)$-models, $\langle \mathfrak{M}, w\rangle$ and $\langle \mathfrak{N}, v\rangle$, are said to be $\mathcal{L}^{\boldsymbol{A}}$-*equivalent* iff for every $\varphi \in L_{\mathcal{L}}(\tau)$, $\|\varphi\|_{\mathcal{L}}^{\langle \mathfrak{M}, w\rangle} = \|\varphi\|_{\mathcal{L}}^{\langle \mathfrak{N}, v\rangle}$.

Lemma 3. *For an arbitrary abstract extension $\mathcal{L}^{\boldsymbol{A}}$ of $\mathcal{L}_0^{\boldsymbol{A}}$ and pointed $(\boldsymbol{A}, \tau)$-models $\langle \mathfrak{M}, w\rangle$ and $\langle \mathfrak{N}, v\rangle$, we have:*

$$Th_{\mathcal{L}}(\mathfrak{M}, w) = Th_{\mathcal{L}}(\mathfrak{N}, v) \Leftrightarrow (\forall \varphi \in L_{\mathcal{L}}(\tau))(\|\varphi\|_{\mathcal{L}}^{\langle \mathfrak{M}, w\rangle} = \|\varphi\|_{\mathcal{L}}^{\langle \mathfrak{N}, v\rangle}).$$

Proof. Note that for any $\varphi \in L_{\mathcal{L}}(\tau)$, any pointed $(\boldsymbol{A}, \tau)$-model $\langle \mathfrak{M}, w\rangle$, and arbitrary $b \in A$, we have $\|\varphi\|_{\mathcal{L}}^{\langle \mathfrak{M}, w\rangle} = b$ iff $\mathfrak{M}, w \models_{\mathcal{L}} (\varphi \to \overline{b}) \wedge (\overline{b} \to \varphi)$. The lemma then follows from the presence of the full set of constants $\overline{b}$ in $\mathsf{Con}^{\boldsymbol{A}}$. □

Definition 3. Take two $(\boldsymbol{A}, \tau)$-models, $\mathfrak{M}$ and $\mathfrak{N}$. A bisimulation between these models is a relation $Z \subseteq W^{\mathfrak{M}} \times W^{\mathfrak{N}}$ such that

(i) If wZv, then $[p]^{\mathfrak{M}}(w) = [p]^{\mathfrak{N}}(v)$.
(ii) If wZv and $R^{\mathfrak{M}}ww'$, then there is $v' \in W_{\mathfrak{N}}$ such that $R^{\mathfrak{N}}vv'$ and $w'Zv'$ (the forth condition).
(iii) If wZv and $R^{\mathfrak{N}}vv'$, then there is $w' \in W_{\mathfrak{M}}$ such that $R^{\mathfrak{M}}ww'$ and $w'Zv'$ (the back condition).

For the pointed models, we will say that $\langle \mathfrak{M}, w\rangle$ and $\langle \mathfrak{N}, v\rangle$ are *bisimilar*, in symbols, $\langle \mathfrak{M}, w\rangle \equiv \langle \mathfrak{N}, v\rangle$ if there is a bisimulation between $\mathfrak{M}$ and $\mathfrak{N}$ relating w to v.

Definition 4. An abstract extension $\mathcal{L}^{\boldsymbol{A}}$ of $\mathcal{L}_0^{\boldsymbol{A}}$ is said to be *strongly invariant for bisimulations* if for all signatures τ and for all $\varphi \in L_{\mathcal{L}}(\tau)$, whenever we have $\langle \mathfrak{M}, w\rangle \equiv \langle \mathfrak{N}, v\rangle$ for the two pointed $(\boldsymbol{A}, \tau)$-models, then $\|\varphi\|_{\mathcal{L}}^{\langle \mathfrak{M}, w\rangle} = \|\varphi\|_{\mathcal{L}}^{\langle \mathfrak{N}, v\rangle}$.

Theorem 4. *$\mathcal{L}_0^{\boldsymbol{A}}$ is strongly invariant for bisimulations.*

See Lemma 3.1 in Marti and Metcalfe [23] for the proof.

Proposition 5. (Hennessy-Milner property for $\mathcal{L}_0^{\boldsymbol{A}}$-saturated models) *In the class of $\mathcal{L}_0^{\boldsymbol{A}}$-saturated $(\boldsymbol{A}, \tau)$-models, $\mathcal{L}_0^{\boldsymbol{A}}$-equivalence implies bisimilarity.*

Proof. Take two $\mathcal{L}_0^{\boldsymbol{A}}$-saturated $\mathfrak{M}$ and $\mathfrak{N}$ such that $\langle \mathfrak{M}, w\rangle$ and $\langle \mathfrak{N}, v\rangle$ are $\mathcal{L}_0^{\boldsymbol{A}}$-equivalent. We claim that the relation Z defined as

$$xZy \Leftrightarrow Th_{\mathcal{L}_0}(\mathfrak{M}, x) = Th_{\mathcal{L}_0}(\mathfrak{N}, y)$$

is a bisimulation. Condition (i) from Definition 3 easily follows from Lemma 3.

For the forth condition, let xZy and $R^{\mathfrak{M}}xx'$. Hence, for every finite $\Phi \subseteq Th_{\mathcal{L}_0}(\mathfrak{M}, x')$ we have $\mathfrak{M}, x \models \Diamond \bigwedge \Phi$, and, by assumption, $\mathfrak{N}, y \models \Diamond \bigwedge \Phi$ so that for some $y_0 \in W^{\mathfrak{N}}$, we must have both $R^{\mathfrak{N}}yy_0$ and $\mathfrak{N}, y_0 \models \bigwedge \Phi$. Hence $Th_{\mathcal{L}_0}(\mathfrak{M}, x')$ is an $(\mathcal{L}_0^{\boldsymbol{A}})_\Diamond$-type of $\langle \mathfrak{N}, y\rangle$, and, by $\mathcal{L}_0^{\boldsymbol{A}}$-saturation of $\mathfrak{N}$, this type is realized by some $R^{\mathfrak{N}}$-successor y' of y. It follows that $Th_{\mathcal{L}_0}(\mathfrak{M}, x') \subseteq Th_{\mathcal{L}_0}(\mathfrak{N}, y')$. In the other direction, assume, for an arbitrary $\varphi \in \mathcal{L}_0$, that $\mathfrak{M}, x' \not\models \varphi$. Then, clearly, $\mathfrak{M}, x' \models \varphi \to \overline{a}$ so that $(\varphi \to \overline{a}) \in Th_{\mathcal{L}_0}(\mathfrak{M}, x') \subseteq Th_{\mathcal{L}_0}(\mathfrak{N}, y')$. But then, of course, $\mathfrak{N}, y' \not\models \varphi$. Since $\varphi \in \mathcal{L}_0$ was chosen arbitrarily, this means that also $Th_{\mathcal{L}_0}(\mathfrak{N}, y') \subseteq Th_{\mathcal{L}_0}(\mathfrak{M}, x')$, whence finally $x'Zy'$, as desired.

The proof of the back condition is entirely analogous except that we use the modal-saturation of $\mathfrak{M}$ instead. □

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4. Tarski union property

Definition 5. Let $\mathfrak{M}$ and $\mathfrak{N}$ be two $(\boldsymbol{A},\tau)$-models and consider an arbitrary abstract extension $\mathcal{L}^{\boldsymbol{A}}$ of $\mathcal{L}_0^{\boldsymbol{A}}$. The function f is said to be an *$\mathcal{L}^{\boldsymbol{A}}$-elementary embedding* from $\mathfrak{M}$ to $\mathfrak{N}$ if f is an embedding and, for every $w \in W^{\mathfrak{M}}$, $Th_{\mathcal{L}}(\mathfrak{M}, w) = Th_{\mathcal{L}}(\mathfrak{N}, f(w))$. Moreover, when f is the identity we obtain the notion of an *$\mathcal{L}^{\boldsymbol{A}}$-elementary substructure*. Again, this group of notions can be trivially extended to pointed models.

When f $\mathcal{L}^{\boldsymbol{A}}$-elementarily embeds $\mathfrak{M}$ into $\mathfrak{N}$, it is readily seen, due to Lemma 3, that f is an isomorphism between $\mathfrak{M}$ and some $\mathcal{L}^{\boldsymbol{A}}$-elementary submodel $\mathfrak{N}'$ of $\mathfrak{N}$. Therefore, by the Isomorphism property of $\mathcal{L}^{\boldsymbol{A}}$, $\mathfrak{M}$ can be identified with $\mathfrak{N}'$ for most of intents and purposes.

A sequence $\{\mathfrak{M}_i \mid i < \omega\}$ of $(\boldsymbol{A},\tau)$-models (where ω is the first infinite ordinal) is called a countable *chain* when for all $i < j < \omega$ we have that $\mathfrak{M}_i$ is a substructure of $\mathfrak{M}_j$. If, moreover, these substructures are $\mathcal{L}$-elementary for some abstract extension $\mathcal{L}^{\boldsymbol{A}}$ of $\mathcal{L}_0^{\boldsymbol{A}}$, we speak of a countable *$\mathcal{L}$-elementary chain*. The *union* of the chain $\{\mathfrak{M}_i \mid i < \omega\}$ is the $(\boldsymbol{A},\tau)$-model $\mathfrak{M} = \bigcup_{i<\omega} \mathfrak{M}_i$ such that $W^{\mathfrak{M}} = \bigcup_{i<\omega} W^{\mathfrak{M}_i}$, $R^{\mathfrak{M}}wv = \sup_{\leq_A}\{R^{\mathfrak{M}_i}wv \mid i < \omega,\ w, v \in W^{\mathfrak{M}_i}\}$, and $[p]^{\mathfrak{M}}(w) = \sup_{\leq_A}\{[p]^{\mathfrak{M}_i}(w) \mid i < \omega,\ w \in W^{\mathfrak{M}_i}\}$ for all $p \in \tau$. Observe as well that $\mathfrak{M}$ is well defined given that $\{\mathfrak{M}_i \mid i < \omega\}$ is a chain.

Definition 6. An abstract extension $\mathcal{L}^{\boldsymbol{A}}$ of $\mathcal{L}_0^{\boldsymbol{A}}$ is said to have *Tarski Union Property* if for all signatures τ, whenever $\{\mathfrak{M}_i \mid i < \omega\}$ is an $\mathcal{L}$-elementary countable chain of $(\boldsymbol{A},\tau)$-models, then $\mathfrak{M}_i$ is an $\mathcal{L}$-elementary substructure of $\mathfrak{M} = \bigcup_{i<\omega} \mathfrak{M}_i$ for every $i < \omega$.

Theorem 6. *$\mathcal{L}_0^{\boldsymbol{A}}$ has Tarski Union Property.*

Proof. We first show that $\mathfrak{M}_i$ is a substructure of $\mathfrak{M}$ for every $i < \omega$. Let $p \in \tau$, let $w \in W^{\mathfrak{M}_i} \cap W^{\mathfrak{M}_j}$ for some $j \in \omega$. Then one of the models $\mathfrak{M}_i, \mathfrak{M}_j$ must be a substructure of the other model, hence $[p]^{\mathfrak{M}_i}(w) = [p]^{\mathfrak{M}_j}(w)$. Since $\mathfrak{M}_j$ was chosen in the chain as an arbitrary model containing w, it follows that $[p]^{\mathfrak{M}_i}(w) = [p]^{\mathfrak{M}}(w)$. The case of R is similar.

Next, we show the $\mathcal{L}$-elementarity, proceeding by induction on the complexity of $\varphi \in \mathcal{L}_{\mathcal{L}_0}(\tau)$. When φ is atomic, the result follows by the fact that $\mathfrak{M}_i$ is a substructure of $\mathfrak{M}$. For any connective $\circ \in \{\wedge, \vee, \to, \&\}$,

$$\|\circ(\psi_1,\psi_2)\|^{\langle\mathfrak{M},w\rangle} = \circ^{\boldsymbol{A}}(\|\psi_1\|^{\langle\mathfrak{M},w\rangle}, \|\psi_2\|^{\langle\mathfrak{M},w\rangle}) = \circ^{\boldsymbol{A}}(\|\psi_1\|^{\langle\mathfrak{M}_i,w\rangle}, \|\psi_2\|^{\langle\mathfrak{M}_i,w\rangle}) = \|\circ(\psi_1,\psi_2)\|^{\langle\mathfrak{M}_i,w\rangle},$$

where the second equality follows by the induction hypothesis and the definition of $\boldsymbol{A}$.

Let $\varphi = \Diamond\psi$ (the case of $\varphi = \Box\psi$ is similar). By induction hypothesis, for every $v \in W^{\mathfrak{M}_i}$, $\|\psi\|^{\langle\mathfrak{M}_i,v\rangle} = \|\psi\|^{\langle\mathfrak{M},v\rangle}$ which implies that

$$\sup_{\leq_A}\{\|\psi\|^{\langle\mathfrak{M}_i,v\rangle} \mid R^{\mathfrak{M}_i}wv\} \leq_{\boldsymbol{A}} \sup_{\leq_A}\{\|\psi\|^{\langle\mathfrak{M},v\rangle} \mid R^{\mathfrak{M}}wv\} = \|\varphi\|^{\langle\mathfrak{M},w\rangle}.$$

On the other hand, to see that

$$\|\varphi\|^{\langle\mathfrak{M},w\rangle} = \sup_{\leq_A}\{\|\psi\|^{\langle\mathfrak{M},v\rangle} \mid R^{\mathfrak{M}}wv\} \leq_{\boldsymbol{A}} \|\varphi\|^{\langle\mathfrak{M}_i,w\rangle},$$

first let $v \in W^{\mathfrak{M}}$ be such that $R^{\mathfrak{M}}wv$ and take a $j \in \omega$ such that v, w are in $W^{\mathfrak{M}_j}$ and we may assume w.l.o.g. that $\mathfrak{M}_j$ is an $\mathcal{L}_0$-elementary extension of $\mathfrak{M}_i$. Then both $R^{\mathfrak{M}_j}wv$ by the $\mathcal{L}_0$-elementary substructure part and $\|\varphi\|^{\langle\mathfrak{M}_j,w\rangle} = \|\varphi\|^{\langle\mathfrak{M}_i,w\rangle}$, whereas $\|\psi\|^{\langle\mathfrak{M}_j,v\rangle} = \|\psi\|^{\langle\mathfrak{M},v\rangle}$ by the induction hypothesis. Using the induction hypothesis, it follows that

$$\|\psi\|^{\langle\mathfrak{M},v\rangle} = \|\psi\|^{\langle\mathfrak{M}_j,v\rangle} \leq_{\boldsymbol{A}} \|\varphi\|^{\langle\mathfrak{M}_j,w\rangle}.$$

Given that the choice of v was arbitrary we obtain the desired inequality. □

5. A first Lindström theorem

This section is devoted to the main theorem of this paper and its proof. It shows that, among its abstract extensions, $\mathcal{L}_0^{\boldsymbol{A}}$ is the $\trianglelefteq$-greatest compact logic which possesses all of the properties established for $\mathcal{L}_0^{\boldsymbol{A}}$ in the preceding sections.

Observe that our Lindström theorems cannot be obtained without the expressive power of having truth constants in our language. By this we mean that our combinations of model-theoretic properties can only characterize languages that are expressively identical to their own expansions with truth constants for all the elements of the algebra. This is because said language expansions also satisfy the model-theoretic properties that we are discussing and, hence, if they increase the expressive power of the original language, the original language cannot be maximal with respect to the properties in question.

We begin by proving some technical statements first. A sequence $\langle w_0, w_1, \ldots, w_n\rangle$ of elements in a model we will denote by $\overrightarrow{w}_n$. Now for a pointed model $\langle\mathfrak{M}, w\rangle$, we can define $\mathfrak{M}w$ as its *unravelling* (and $\langle\mathfrak{M}w, \langle w\rangle\rangle$ as its *pointed unravelling*) in the usual way, except that now we define the $\boldsymbol{A}$-valued function $[p]^{\mathfrak{M}w}$ (corresponding to a propositional variable p) as $[p]^{\mathfrak{M}w}(\overrightarrow{v}_n) = [p]^{\mathfrak{M}}(v_n)$ for every sequence of worlds with $w = v_0$. It is easy to check that relating every such $\overrightarrow{v}_n$ to v_n provides a bisimulation between $\langle\mathfrak{M}w, \langle w\rangle\rangle$ and $\langle\mathfrak{M}, w\rangle$. An $(\boldsymbol{A}, \tau)$-model $\mathfrak{N}$ we call an unravelled model iff $\mathfrak{N} = \mathfrak{M}w$ for some $\langle\mathfrak{M}, w\rangle$.

Proposition 7. *Let $\mathcal{L}^{\boldsymbol{A}}$ be an abstract extension of $\mathcal{L}_0^{\boldsymbol{A}}$ and $\langle\mathfrak{M}, w\rangle$ an arbitrary $(\boldsymbol{A}, \tau)$-model. If $\mathcal{L}^{\boldsymbol{A}}$ has the Compactness property and is strongly invariant for bisimulations, then for any $w \in W^{\mathfrak{M}}$ the unravelling $\mathfrak{M}w$ is an $\mathcal{L}^{\boldsymbol{A}}$-elementary substructure of some unravelled $(\boldsymbol{A}, \tau)$-model $\mathfrak{N}$ such that, for every $\overrightarrow{v}_n \in W^{\mathfrak{M}w}$ and every $\mathcal{L}_\Diamond^{\boldsymbol{A}}$-type Φ of $\langle\mathfrak{M}w, \overrightarrow{v}_n\rangle$, Φ is satisfied by some $u \in W^{\mathfrak{N}}$ such that $R^{\mathfrak{N}}\overrightarrow{v}_n u$.*

Proof. First consider the signature τ' which is the expansion of τ by adding the following disjoint collections of new predicates:

$$\{P_{\overrightarrow{v}_n} \mid \overrightarrow{v}_n \in W^{\mathfrak{M}w}\},$$
$$\{Q_{\overrightarrow{v}_n,\Phi} \mid \overrightarrow{v}_n \in W^{\mathfrak{M}w}, \Phi \text{ is a } \mathcal{L}_\Diamond^{\boldsymbol{A}}\text{-type of } \langle\mathfrak{M}w, \overrightarrow{v}_n\rangle\}.$$

Now we can consider the theory Θ in the signature τ' which contains all the following formulas (where $\Diamond^n$, $\Box^n$ abbreviates n many iterations of the symbols $\Diamond$, $\Box$ respectively):

$$\Diamond^{n+1} P_{\overrightarrow{v}_n} \qquad \overrightarrow{v}_n \in W^{\mathfrak{M}w}, n \in \omega$$
$$\Box^k((P_{\overrightarrow{v}_n} \to \overline{0}) \vee P_{\overrightarrow{v}_n}) \qquad \overrightarrow{v}_n \in W^{\mathfrak{M}w}, k, n \in \omega$$
$$\Box^k((P_{\overrightarrow{v}_n} \to \overline{0}) \vee (P_{\overrightarrow{u}_r} \to \overline{0})) \qquad \overrightarrow{v}_n \neq \overrightarrow{u}_r, k, n, r \in \omega$$
$$\Box^k((P_{\overrightarrow{v}_n} \to \overline{0}) \vee \Diamond P_{\overrightarrow{v}_r}) \qquad R^{\mathfrak{M}w}(\overrightarrow{v}_n, \overrightarrow{v}_r) = \overline{1}^{\boldsymbol{A}}, k, n, r \in \omega$$
$$\Box^k((P_{\overrightarrow{v}_n} \to \overline{0}) \vee \Box(P_{\overrightarrow{v}_r} \to \overline{0})) \qquad R^{\mathfrak{M}w}(\overrightarrow{v}_n, \overrightarrow{v}_r) = \overline{0}^{\boldsymbol{A}}, k, n, r \in \omega$$
$$\Box^k((P_{\overrightarrow{v}_n} \to \overline{0}) \vee \theta) \qquad \theta \in Th_{\mathcal{L}}(\mathfrak{M}w, \overrightarrow{v}_n), k, n \in \omega$$
$$\Box^k((Q_{\overrightarrow{v}_n,\Phi} \to \overline{0}) \vee Q_{\overrightarrow{v}_n,\Phi}) \qquad \Phi \text{ is a } \mathcal{L}_\Diamond^{\boldsymbol{A}}\text{-type of } \langle\mathfrak{M}w, \overrightarrow{v}_n\rangle, k, n \in \omega$$
$$\Box^k((P_{\overrightarrow{v}_n} \to \overline{0}) \vee \Diamond Q_{\overrightarrow{v}_n,\Phi}) \qquad \Phi \text{ is a } \mathcal{L}_\Diamond^{\boldsymbol{A}}\text{-type of } \langle\mathfrak{M}w, \overrightarrow{v}_n\rangle, k, n \in \omega$$
$$\Box^k((Q_{\overrightarrow{v}_n,\Phi} \to \overline{0}) \vee \varphi) \qquad \Phi \text{ is a } \mathcal{L}_\Diamond^{\boldsymbol{A}}\text{-type of } \langle\mathfrak{M}w, \overrightarrow{v}_n\rangle, \varphi \in \Phi, k, n \in \omega$$

Every finite subset Θ_0 of this theory has a model since we can expand $\mathfrak{M}w$ to a model of Θ_0 by letting $P_{\overrightarrow{v}_n}$ be the crisp set containing just $\overrightarrow{v}_n$, whereas $Q_{\overrightarrow{v}_n,\Phi}$ will be the crisp set containing all worlds satisfying every member of Φ appearing in Θ_0 among successors of v_n. Then, by Compactness, there is a pointed $(\boldsymbol{A}, \tau')$-model $\langle\mathfrak{M}_0, v\rangle$ of Θ. We then consider the τ-reduct $\mathfrak{M}_0|\tau$ of $\mathfrak{M}_0$ and use strong invariance for bisimulation to set $\mathfrak{N} := (\mathfrak{M}_0|\tau)v$. We can further define an $\mathcal{L}^{\boldsymbol{A}}$-elementary embedding f of $\langle\mathfrak{M}w, \langle w\rangle\rangle$ into $\langle\mathfrak{N}, \langle v\rangle\rangle$ by induction on the length n of $\overrightarrow{v}_n \in W^{\mathfrak{M}w}$. We construct it in such a way that for every $\overrightarrow{u}_k \in W^{\mathfrak{N}}$, whenever $f(\overrightarrow{v}_r) = \overrightarrow{u}_k$, then $k = r$ and $\mathfrak{M}_0, u_k \models_{\mathcal{L}} P_{\overrightarrow{v}_r}$. For $n = 1$, we set $f(\langle w\rangle) = \langle v\rangle$, and for $n = k+1$ we assume that $f(\overrightarrow{v}_k)$ is defined and that $\overrightarrow{v}_{k+1} \in W^{\mathfrak{M}w}$. We then use the axiom of choice to ensure a choice of a $\langle f(\overrightarrow{v}_k), u\rangle \in W^{\mathfrak{N}}$ such that $\mathfrak{M}_0, u \models_{\mathcal{L}} P_{\overrightarrow{v}_{k+1}}$. Existence of such u is guaranteed by the induction hypothesis and the definition of Θ. As for injectivity, assume that

Please cite this article in press as: G. Badia, G. Olkhovikov, A Lindström theorem in many-valued modal logic over a finite MTL-chain, Fuzzy Sets Syst. (2019), https://doi.org/10.1016/j.fss.2019.03.002

for some $\overrightarrow{v}_n, \overrightarrow{u}_r \in W^{\mathfrak{M}w}$ such that $\overrightarrow{v}_n \neq \overrightarrow{u}_r$, we have $f(\overrightarrow{v}_n) = f(\overrightarrow{u}_r) = \overrightarrow{t}_k \in W^{\mathfrak{N}}$. Then, since we have, by definition of Θ, that $\mathfrak{M}_0, \langle v \rangle \models_{\mathcal{L}} \Box^k((P_{\overrightarrow{v}_n} \to \overline{0}) \vee (P_{\overrightarrow{u}_r} \to \overline{0}))$, it follows that $\mathfrak{M}_0, \overrightarrow{t}_k \models_{\mathcal{L}} (P_{\overrightarrow{v}_n} \to \overline{0}) \vee (P_{\overrightarrow{u}_r} \to \overline{0})$. On the other hand, by $f(\overrightarrow{v}_n) = f(\overrightarrow{u}_r) = \overrightarrow{t}_k$, we must have both $\mathfrak{M}_0, \overrightarrow{t}_k \models_{\mathcal{L}} P_{\overrightarrow{v}_n}$ and $\mathfrak{M}_0, \overrightarrow{t}_k \models_{\mathcal{L}} P_{\overrightarrow{u}_r}$, which immediately leads to a contradiction showing that f must be injective. The preservation of theories and the realization of types then follow from the description of Θ and the strong invariance of $\mathcal{L}^{\boldsymbol{A}}$ for bisimulation. Since $\langle \mathfrak{M}w, \langle w \rangle\rangle$ is thus shown to be $\mathcal{L}^{\boldsymbol{A}}$-elementarily embeddable into $\langle \mathfrak{N}, \langle v \rangle\rangle$, it is isomorphic to an $\mathcal{L}^{\boldsymbol{A}}$-elementary submodel of $\langle \mathfrak{N}, \langle v \rangle\rangle$ and, due to Isomorphism property, can be treated as such. □

Proposition 8. *Let $\mathcal{L}^{\boldsymbol{A}}$ be an abstract extension of $\mathcal{L}_0^{\boldsymbol{A}}$. If $\mathcal{L}^{\boldsymbol{A}}$ is compact, has Tarski Union Property, and is strongly invariant for bisimulations, then, for every pointed $(\boldsymbol{A}, \tau)$-model $\langle \mathfrak{M}, w \rangle$, its unravelling $\mathfrak{M}w$ can be $\mathcal{L}^{\boldsymbol{A}}$-elementarily extended to an $\mathcal{L}^{\boldsymbol{A}}$-saturated model.*

Proof. Setting $\mathfrak{M}w := \mathfrak{M}_0$, we build a countable $\mathcal{L}^{\boldsymbol{A}}$-elementary chain $\{\mathfrak{M}_i \mid i < \omega\}$ to get the $\mathcal{L}^{\boldsymbol{A}}$-saturated extension $\mathfrak{N} = \bigcup\{\mathfrak{M}_i \mid i < \omega\}$ by Tarski Union Property. For every $i \in \omega$ we set as $\mathfrak{M}_{i+1}$ the unravelled $(\boldsymbol{A}, \tau)$-model which $\mathcal{L}^{\boldsymbol{A}}$-elementarily extends $\mathfrak{M}_i$ while realizing every $\mathcal{L}^{\boldsymbol{A}}_{\Diamond}$-type of every pointed model based on $\mathfrak{M}_i$. The existence of such an $\mathfrak{M}_{i+1}$ is guaranteed by Proposition 7.

It remains to show that $\mathfrak{N}$ is $\mathcal{L}^{\boldsymbol{A}}$-saturated. Suppose that Φ is an arbitrary $\mathcal{L}^{\boldsymbol{A}}_{\Diamond}$-type of $\langle \mathfrak{N}, v \rangle$ for some $v \in W^{\mathfrak{N}}$. Then $v \in W_{\mathfrak{M}_i}$ for some $i \in \omega$. But then, for every finite $\Phi_0 \subseteq \Phi$ we will have $\Diamond \bigwedge \Phi_0 \in Th_{\mathcal{L}}(\mathfrak{M}, v) = Th_{\mathcal{L}}(\mathfrak{M}_i, v)$, where the equality of theories holds by Tarski Union Property. Therefore, Φ is an $\mathcal{L}^{\boldsymbol{A}}_{\Diamond}$-type of $\langle \mathfrak{M}_i, v \rangle$ and hence for some $u \in W^{\mathfrak{M}_{i+1}}$ such that $R^{\mathfrak{M}_{i+1}}vu$ we must have $\Phi \subseteq Th_{\mathcal{L}}(\mathfrak{M}_{i+1}, u) = Th_{\mathcal{L}}(\mathfrak{M}, u)$ by the construction of our chain. It remains to notice that we must have $R^{\mathfrak{N}}vu$ so that Φ is realized in $\mathfrak{N}$. □

Theorem 9. *Let $\mathcal{L}^{\boldsymbol{A}}$ be an abstract logic such that $\mathcal{L}_0^{\boldsymbol{A}} \trianglelefteq \mathcal{L}^{\boldsymbol{A}}$. If $\mathcal{L}^{\boldsymbol{A}}$ is compact, has Tarski Union Property, and is strongly invariant for bisimulations, then $\mathcal{L}^{\boldsymbol{A}} \simeq \mathcal{L}_0^{\boldsymbol{A}}$.*

Proof. Assume for reduction that $\mathcal{L}^{\boldsymbol{A}} \ntrianglelefteq \mathcal{L}_0^{\boldsymbol{A}}$, so we can find, by Occurrence property, a $\varphi \in \mathcal{L}_{\mathcal{L}}(\tau_\varphi)$ that is not 1-equivalent to any formula in $\mathcal{L}_0(\tau_\varphi)$. We enumerate the countable set $\mathcal{L}_0(\tau_\varphi)$ as $\psi_1, \psi_2, \ldots$ and then we define a list $(\varphi_i)_{i \in \omega}$ of formulas in $\mathcal{L}_{\mathcal{L}}(\tau_\varphi)$ such that for any n,

$$\bigwedge_{i=0}^{n} \varphi_i$$

is not 1-equivalent to any formula in $\mathcal{L}_0(\tau_\varphi)$ as follows. First, $\varphi_0 := \varphi$. Now, suppose that

$$\bigwedge_{i=0}^{k} \varphi_i$$

is not 1-equivalent to any formula in $\mathcal{L}_0(\tau_\varphi)$. Recall that we have $a \in A$ as the immediate predecessor of $1^{\boldsymbol{A}}$ in terms of $\leq_{\boldsymbol{A}}$. We have either

$$(\bigwedge_{i=0}^{k} \varphi_i) \wedge \psi_{k+1}$$

or

$$(\bigwedge_{i=0}^{k} \varphi_i) \wedge (\psi_{k+1} \to \overline{a})$$

is not 1-equivalent to any formula in $\mathcal{L}_0(\tau_\varphi)$. For otherwise,

$$(\bigwedge_{i=0}^{k} \varphi_i) \wedge \psi_{k+1}$$

is 1-equivalent to a $\theta_0 \in \mathcal{L}_0(\tau_\varphi)$, and

$$(\bigwedge_{i=0}^{k} \varphi_i) \wedge (\psi_{k+1} \to \overline{a})$$

is 1-equivalent to a $\theta_1 \in \mathcal{L}_0(\tau_\varphi)$. But $\psi_{k+1} \vee (\psi_{k+1} \to \overline{a})$ is satisfied in every $\mathscr{L}$-model. Hence, for any arbitrary model $\langle \mathfrak{N}, v \rangle$

$$\mathfrak{N}, v \models_{\mathscr{L}} \bigwedge_{i=0}^{k} \varphi_i \text{ only if } \mathfrak{N}, v \models_{\mathscr{L}} \theta_0 \vee \theta_1,$$

and, moreover,

$$\mathfrak{N}, v \models_{\mathscr{L}} \theta_0 \vee \theta_1 \text{ only if } \mathfrak{N}, v \models_{\mathscr{L}} \bigwedge_{i=0}^{k} \varphi_i.$$

Then we would have that

$$\bigwedge_{i=0}^{k} \varphi_i$$

is 1-equivalent to a formula in $\mathcal{L}_0(\tau_\varphi)$, which is a contradiction. Finally let $\varphi_{k+1} = \psi_{k+1}$ or $\varphi_{k+1} = (\psi_{k+1} \to \overline{a})$ according to which alternative holds (if both, then make an arbitrary choice).

We observe that

$$(\varphi \to \overline{a}) \wedge (\bigwedge_{i=1}^{n} \varphi_i)$$

is satisfiable. Otherwise, every model $\langle \mathfrak{M}, w \rangle$ of $(\bigwedge_{i=1}^{n} \varphi_i)$ would be one in which $\|\varphi\|_{\mathscr{L}}^{\langle \mathfrak{M}, w \rangle} \not\leq_{\boldsymbol{A}} a$, and, by linearity, $a <_{\boldsymbol{A}} \|\varphi\|_{\mathscr{L}}^{\langle \mathfrak{M}, w \rangle}$ so that $\overline{1}^{\boldsymbol{A}} = \|\varphi\|_{\mathscr{L}}^{\langle \mathfrak{M}, w \rangle}$, that is, $\mathfrak{M}, w \models_{\mathscr{L}} \varphi$. And, hence, $\varphi \wedge (\bigwedge_{i=1}^{n} \varphi_i)$ is 1-equivalent to $(\bigwedge_{i=1}^{n} \varphi_i)$, which is a contradiction.

Then for any n,

$$\varphi \wedge (\bigwedge_{i=1}^{n} \varphi_i) \text{ and } (\varphi \to \overline{a}) \wedge (\bigwedge_{i=1}^{n} \varphi_i)$$

both have models. For otherwise, one of them would be 1-equivalent to $\overline{0}^{\boldsymbol{A}}$. Hence, by Compactness, we can obtain models $\langle \mathfrak{M}, w \rangle$ and $\langle \mathfrak{N}, v \rangle$ in the signature τ of the sets $\{\varphi_i \mid 1 \leq i\} \cup \{\varphi\}$ and $\{\varphi_i \mid 1 \leq i\} \cup \{\varphi \to \overline{a}\}$ respectively. Furthermore, by Lemma 3, $\langle \mathfrak{M}, w \rangle$ and $\langle \mathfrak{N}, v \rangle$ coincide on the value for every formula in $\mathcal{L}_0(\tau_\varphi)$.

By strong invariance for bisimulation, we may take the unravelings $\mathfrak{M}w$ and $\mathfrak{N}v$ of our two models. Hence, in the presence of the TUP, we can extend $\mathfrak{M}w$ and $\mathfrak{N}v$ to $\mathcal{L}_0^{\boldsymbol{A}}$-saturated elementary extensions $\mathfrak{M}'$ and $\mathfrak{N}'$. By Closure property it follows that $\mathfrak{M}'$ and $\mathfrak{N}'$ are also $\mathscr{L}^{\boldsymbol{A}}$-saturated. Further, by the Hennessy-Milner theorem, $\langle \mathfrak{M}', \langle w \rangle \rangle$ and $\langle \mathfrak{N}', \langle v \rangle \rangle$ are bisimilar, but they differ on the value for φ, which contradicts the strong invariance of $\mathscr{L}^{\boldsymbol{A}}$ for bisimulation. □

6. A second Lindström theorem

In this section, we will outline how to replace the TUP with a relativization property along the lines of van Benthem [26]. The modal degree of a formula is defined in the standard way, measuring the number of nested modal operators in a formula.

Lemma 10. *Let $\mathscr{L}^{\boldsymbol{A}}$ be a compact logic extending $\mathcal{L}_0^{\boldsymbol{A}}$. Suppose φ is a formula of $\mathscr{L}^{\boldsymbol{A}}$ which is preserved under $\mathcal{L}_0^{\boldsymbol{A}}$-equivalence up to modal degree k, then φ is 1-equivalent to a formula of $\mathcal{L}_0^{\boldsymbol{A}}$ of degree k.*

Proof. φ is 1-equivalent to the infinitary formula

$$\bigvee_{\mathfrak{M},w\models\varphi} \bigwedge Th^k_{\mathcal{L}_0}(\mathfrak{M}, w).$$

Compactness of $\mathcal{L}^{\boldsymbol{A}}$ allows us to use Corollary 2 to bring down the latter formula to a finitary formula of $\mathcal{L}^{\boldsymbol{A}}_0$. First, observe that if $\mathfrak{M}, w \models \varphi$, then $\bigwedge Th^k_{\mathcal{L}_0}(\mathfrak{M}, w)$ 1-implies φ, since φ is preserved under $\mathcal{L}^{\boldsymbol{A}}_0$-equivalence up to modal degree k. By compactness then, $\bigwedge Th^{k\prime}_{\mathcal{L}_0}(\mathfrak{M}, w)$ 1-implies φ for some finite $Th^{k\prime}_{\mathcal{L}_0} \subseteq Th^k_{\mathcal{L}_0}$. Then φ is 1-equivalent to the infinitary formula

$$\bigvee_{\mathfrak{M},w\models\varphi} \bigwedge Th^{k\prime}_{\mathcal{L}_0}(\mathfrak{M}, w).$$

But then there is no model where φ takes value $1^{\boldsymbol{A}}$ while every $\bigwedge Th^{k\prime}_{\mathcal{L}_0}(\mathfrak{M}, w)$ takes a value $< 1^{\boldsymbol{A}}$. By compactness, this must be so for some finite collection Θ of the formulas $\bigwedge Th^{k\prime}_{\mathcal{L}_0}(\mathfrak{M}, w)$. But then φ is 1-equivalent to the finitary formula

$$\bigvee_{Th^{k\prime}_{\mathcal{L}_0}(\mathfrak{M},w)\in\Theta} \bigwedge Th^{k\prime}_{\mathcal{L}_0}(\mathfrak{M}, w). \quad \square$$

The *finite depth property* for an abstract logic says that for any formula φ, there is a natural k such that, for all models $\langle\mathfrak{M}, w\rangle$,

$$\langle\mathfrak{M}, w\rangle \models \varphi \text{ iff } \langle\mathfrak{M}, w\rangle|k \models \varphi,$$

where $\langle\mathfrak{M}, w\rangle|k$ is the submodel of $\langle\mathfrak{M}, w\rangle$ with the worlds that can be reached from w in $\leq k$ many steps. It is not hard to prove that $\mathcal{L}^{\boldsymbol{A}}_0$ has this property using the notion of the modal degree of a formula.

Lemma 11. *Assume that $\mathcal{L}^{\boldsymbol{A}}$ is a logic extending $\mathcal{L}^{\boldsymbol{A}}_0$ and is strongly invariant for bisimulation. Suppose φ is a formula of $\mathcal{L}^{\boldsymbol{A}}$ which has the finite depth property for some k, then φ is preserved under $\mathcal{L}^{\boldsymbol{A}}_0$-equivalence up to modal degree k.*

Proof. Let models $\langle\mathfrak{M}, w\rangle$ and $\langle\mathfrak{N}, v\rangle$ be such that $Th^k_{\mathcal{L}_0}(\mathfrak{M}, w) = Th^k_{\mathcal{L}_0}(\mathfrak{N}, v)$. Suppose that $\mathfrak{M}, w \models \varphi \leftrightarrow \overline{b}$ for some $b \in \boldsymbol{A}$. We can take the bisimilar unravelings $\mathfrak{M}w$ and $\mathfrak{N}v$ of our models and notice that there is a natural bisimulation between the restriction of the models to depth k, $\mathfrak{M}w|k$ and $\mathfrak{N}v|k$. Using the finite depth property, we have that $\mathfrak{M}w|k \models \varphi \leftrightarrow \overline{b}$, and by strong invariance for bisimulation, $\mathfrak{N}v|k \models \varphi \leftrightarrow \overline{b}$, and by the finite depth property again, $\mathfrak{N}v \models \varphi \leftrightarrow \overline{b}$. $\square$

From the previous two lemmas we can easily establish the following.

Corollary 12. *Let $\mathcal{L}^{\boldsymbol{A}}$ be an abstract logic such that $\mathcal{L}^{\boldsymbol{A}}_0 \trianglelefteq \mathcal{L}^{\boldsymbol{A}}$. If $\mathcal{L}^{\boldsymbol{A}}$ is compact, has the finite depth property, and is strongly invariant for bisimulations, then $\mathcal{L}^{\boldsymbol{A}} \simeq \mathcal{L}^{\boldsymbol{A}}_0$.*

Given a crisp propositional variable P in a given structure $\langle\mathfrak{M}, w\rangle$ (i.e., $((P \to \overline{0}) \vee P)$ holds in every v finitely reachable from w) and a modal formula φ we define the relativization of φ to P in $\mathcal{L}^{\boldsymbol{A}}_0$, in symbols φ^P, inductively as follows:

- φ^P is just φ if φ is atomic.
- $\varphi^P = \circ(\psi_0^P, \dots, \psi_n^P)$ when $\varphi = \circ(\psi_0, \dots, \psi_n)$ for some connective $\circ$.
- $\varphi^P = \Diamond(P \wedge \psi^P)$ if $\varphi = \Diamond\psi$.
- $\varphi^P = \Box((P \to \overline{0}) \vee \psi^P)$ if $\varphi = \Box\psi$.

It is not difficult to see that, for any model $\langle \mathfrak{M}, w\rangle$, if $\langle \mathfrak{M}, w\rangle \models P$, then $\langle \mathfrak{M}, w\rangle \models \varphi^P$ iff $\langle \mathfrak{M}, w\rangle | P \models \varphi$, where the latter is the submodel of $\langle \mathfrak{M}, w\rangle$ containing just the worlds where P takes value $1^{\boldsymbol{A}}$. The *relativization property* is simply the generalization of this fact to an arbitrary extension of $\mathcal{L}_0^{\boldsymbol{A}}$.

Theorem 13. *Let $\mathcal{L}^{\boldsymbol{A}}$ be an abstract logic such that $\mathcal{L}_0^{\boldsymbol{A}} \trianglelefteq \mathcal{L}^{\boldsymbol{A}}$. If $\mathcal{L}^{\boldsymbol{A}}$ is compact, has the relativization property, and is strongly invariant for bisimulations, then $\mathcal{L}^{\boldsymbol{A}} \simeq \mathcal{L}_0^{\boldsymbol{A}}$.*

Proof. We prove this result from Corollary 12. So suppose that $\mathcal{L}^{\boldsymbol{A}}$ does not have the finite depth property to derive a contradiction. Take φ such that for every k there is a model $\langle \mathfrak{M}_k, w\rangle$ such that $\mathfrak{M}_k, w \models \varphi$ while $\mathfrak{M}_k|k, w \not\models \varphi$. Let $b \in \boldsymbol{A}$ denote the immediate predecessor of $1^{\boldsymbol{A}}$. Introduce next a new propositional variable P and consider the collection Θ of formulas:

$$\varphi, (\varphi \to \overline{b})^P, \Box^n P \ (n \in \omega)$$

This set is finitely satisfiable, so the whole thing must have a model $\langle \mathfrak{M}, w\rangle$. Take the generated submodel $\langle \mathfrak{M}', w\rangle$ containing w and all points finitely reachable from it in $\mathfrak{M}$. By strong invariance for bisimulation, $\langle \mathfrak{M}', w\rangle \models \varphi$ and, by the rest of Θ holding, we also conclude that $\langle \mathfrak{M}', w\rangle \not\models \varphi$, which is a contradiction. $\Box$

It would be very interesting to strengthen our results to equivalence in the sense of coincidence for every value, not just 1-equivalence, however, we do not know how to do this.

7. Final remarks

As we mentioned above, our main result follows the template of Piro and Otto [24]. The similarities mainly concern the use of Tarski Union Property, which proved to be a powerful tool in other similar contexts (see, e.g. Enqvist [16] and Zoghifard and Pourmahdian [28]). Moreover, there was also the possibility of strengthening Theorem 9 by omitting from it any mentions of this property. This required switching to a version of the proof given in van Benthem [26].

Acknowledgements

We are grateful to two anonymous referees and the editor of this journal for their numerous and helpful comments. Their help greatly improved the paper. Guillermo Badia is supported by the project I 1923-N25 of the Austrian Science Fund (FWF). Grigory Olkhovikov is supported by Deutsche Forschungsgemeinschaft (DFG), project WA 936/11-1.

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Please cite this article in press as: G. Badia, G. Olkhovikov, A Lindström theorem in many-valued modal logic over a finite MTL-chain, Fuzzy Sets Syst. (2019), https://doi.org/10.1016/j.fss.2019.03.002

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